

ON THE CONVEXITY OF NUMERICAL RANGE OVER CERTAIN FIELDS

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ABSTRACT. Let L be a degree 2 Galois extension of the field K and M an $n \times n$ matrix with coefficients in L . Let $\langle \cdot, \cdot \rangle : L^n \times L^n \rightarrow L$ be the sesquilinear form associated to the involution $\sigma : L \rightarrow L$ fixing K . This sesquilinear form defines the numerical range $\text{Num}(M)$ of any $n \times n$ matrix over L . In this paper we study the convexity of $\text{Num}(M)$ (under certain assumptions on K and/or M). Many of the results are for ordered fields.

1. INTRODUCTION

For any field F let $M_{n,n}(F)$ denote the set of all $n \times n$ matrices with coefficients in F . Fix fields $K \subset L$ such that L is a degree 2 Galois extension of K and call σ the generator of the Galois group of the extension $K \subset L$. For any $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in L^n$ set $\langle u, v \rangle := \sum_{i=1}^n \sigma(u_i)v_i$. The map $\langle \cdot, \cdot \rangle : L^n \times L^n \rightarrow L$ is sesquilinear (linear in the second variable and σ -linear in the first one) and $\langle u, v \rangle = \sigma(\langle v, u \rangle)$ for all $u, v \in L^n$. For any $M \in M_{n,n}(L)$ and $u \in L^n$ set $\nu_M(u) := \langle u, Mu \rangle$. We obtain a map $\nu_M : L^n \rightarrow L$, call the *numerical map* of M . When $K = \mathbb{R}$ and $L = \mathbb{C}$ (and so σ is the complex conjugation and $\langle \cdot, \cdot \rangle$ is the usual Hermitian product) the image of the restriction of ν_M to $C_n(1) := \{z \in L^n \mid \langle z, z \rangle = 1\}$ is called the *numerical range* of M ([5], [9]); it is always a convex subset of \mathbb{C} ([5], [9]) and the main aim of this paper is to explore the convexity for other (L, K, σ) , sometimes with strong restrictions on the matrix M . Under mild assumptions on L and K we have $M = M^\dagger$ if and only if $\text{Num}(M) \subseteq K$ (Proposition 3), but this is not always true (Example 1). Fix an integer $k > 0$ and $M_1, \dots, M_k \in M_{n,n}(L)$. Let $\nu_{M_1, \dots, M_k} : L^n \rightarrow L^k$ be the map defined by the formula $\nu_{M_1, \dots, M_k}(u) = (\langle u, M_1 u \rangle, \dots, \langle u, M_k u \rangle)$; we call it the *joint numerical map* of M_1, \dots, M_k . When $L = \mathbb{C}$ the image of the map $\nu_{M_1, \dots, M_k}|_{C_n(1)}$ is called the *joint numerical range* of the matrices M_1, \dots, M_k ([8]). In [2] we introduced the following subsets of K . Let $\Delta \subseteq L$ be the set of all $\langle a, a \rangle$, $a \in L$. Since $\sigma(\langle a, a \rangle) = \langle a, a \rangle$ for any $a \in L$, we have $\Delta \subseteq K$. Note that $0 \in \Delta$, that $\Delta \setminus \{0\}$ is a multiplicative group and that Δ is the image of the norm map $L \rightarrow K$. Since $\langle a, a \rangle = a^2$ for all $a \in K$, Δ contains all squares of elements of K . For each $n \geq 1$ let $\Delta_n \subseteq K$ be the sum on n elements of Δ . The set Δ_n is the set of all $\langle u, u \rangle$ for some $u \in L^n$. We have $\Delta_n + \Delta_m = \Delta_{n+m}$ for all $n > 0, m > 0$ and it is easy to check that $\Delta_n \setminus \{0\}$ is a multiplicative group ([2, Lemma 2]).

Take $a, b \in L$ such that $a \neq b$. The Δ -convex hull of $\{a, b\}$ is the set of all $ta + (1-t)b$ with $t \in \Delta \cap (1 - \Delta)$. At least if $\text{char}(K) = 0$, $\Delta \cap (1 - \Delta)$ is infinite

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([2, Lemma 8]). A set $S \subseteq L$ is said to be Δ -convex if for all $a, b \in S$ with $a \neq b$, S contains the Δ -convex hull of $\{a, b\}$. For any $S \subseteq L$ the Δ -convex hull of S is the intersection of all Δ -convex subsets of L containing S , i.e. the smallest Δ -convex subsets of L containing S . If we take $\Delta_n \cap (1 - \Delta_n)$ instead of $\Delta \cap (1 - \Delta)$ we get the notion of Δ_n -convexity.

We met the following obstacles to get the convexity of $\text{Num}(M)$ and to study the numerical range, the joint numerical range, the numerical map and the joint numerical map.

- (1) L may be not algebraically closed and hence there are matrices $M \in M_{n,n}(L)$, $n \geq 2$, with eigenvalues not contained in L ;
- (2) there are $u \in L^n$, $n \geq 2$, with $u \neq 0$ and $\langle u, u \rangle = 0$;
- (3) there are $u \in L^n$ such that $\langle u, u \rangle \neq 0$, but there is no $t \in L$ with $\langle tu, tu \rangle = 1$;
- (4) Δ_n may strictly contain Δ .

The obstacles (3) and (4) are equivalent (Remark 1). It is very restrictive to assume that L is algebraically closed, but the only case in which we get Δ -convexity of all numerical ranges (Theorem 2) requires that K is a real closed field and so L is algebraically closed ([3, Theorem 1.2.2]). When L is not algebraically closed we may at least get informations for some matrices, e.g. the ones with all eigenvalues contained in L , plus some other conditions are satisfied. We discussed (2), (3) and (4) in [2]. Here we describe another case in which $\Delta_n = \Delta$ for all n and $\langle u, u \rangle = 0$ for some $u \in L^n$ if and only if $u = 0$ (Proposition 1). We say that $\langle \cdot, \cdot \rangle$ is *definite up to n* if $\langle u, u \rangle \neq 0$ for all $u \in L^n \setminus \{0\}$.

When L and K are finite fields, say $K = \mathbb{F}_q$, $L = \mathbb{F}_{q^2}$ and σ the Frobenius map $t \mapsto t^q$, we have $\Delta = K$ and so $\Delta_n = \Delta$ for all $n > 0$, so the third condition is always satisfied, but the only non-empty Δ -convex subsets of L are the singletons, L and the affine K -lines of L seen as a 2-dimensional K vector space; there are plenty of examples ([4]) of matrices M with $\#(\text{Num}(M)) \notin \{1, q, q^2\}$ and hence with numerical range not Δ -convex. The obstacle (2) may be overcome in all cases in which we have some definite positive conditions (e.g. if K is a subfield of \mathbb{R} and $L = K(i)$ with σ the complex conjugation). We axiomatize the non-existence of obstacle (2) in the following way. We say that $\langle \cdot, \cdot \rangle$ is *definite of to n* if $\langle u, u \rangle \neq 0$ for all $u \in L^n$ with $u \neq 0$. In the set-up of (2) and (3) even if M has an eigenvalue $c \in L$ we may have $c \notin \text{Num}(M)$ (e.g. see Example 2; there are many examples over finite fields ([4])).

The classical text-book proof of the Toeplitz-Hausdorff theorem on the convexity of $\text{Num}(M)$ uses a reduction to the case in which the numerical range is either a segment or a region bounded by an ellipse with one or two foci ([5, §1.1]). We axiomatize this case in the following way.

Definition 1. A Δ_n -ellipse with a unique focus at 0 is a subset $S \subseteq L$ such that there are $\delta_1 \in \Delta_n \setminus \{0\}$ and $\delta_2 \in \Delta_n \setminus \{0\}$ with $S = \{\sigma(x)y \mid (x, y) \in L^2, \delta_1 x \sigma(x) + \delta_2 y \sigma(y) = 1\}$. In this case we say that S is a Δ_n -ellipse with a unique focus with parameters (δ_1, δ_2) . A Δ_n -ellipse with two foci is a subset $S \subseteq L$ such that there are $\delta_1, \delta_2 \in \Delta_n \setminus \{0\}$ and $d_1, d_2 \in L^*$ with $S = \{d_1 y \sigma(x) + d_2 y \sigma(y) \mid (x, y) \in L^2, \delta_1 x \sigma(x) + \delta_2 y \sigma(y) = 1\}$; $(\delta_1, \delta_2, d_1, d_2)$ are the parameters of the ellipse with 2 foci. A Δ_n -ellipse is set $S \subseteq L$ such that there are $a, b \in L$ with $b \neq 0$ and $(S - a)/b$ is a Δ_n -ellipse with one or two foci. We define in the same way the Δ -ellipses.

When K is a finite field, a Δ -ellipse is not Δ -convex. When $n = 2$ we find some cases in which $\text{Num}(M)$ is an ellipse with one or two foci (Propositions 4, 5). We show some cases in which for any $a, b \in \text{Num}(M)$ the set $\text{Num}(M)$ contains either the Δ -convex hull of $\{a, b\}$ or a Δ -ellipse with 2 foci containing $\{a, b\}$ (Propositions 4, 5 and 2). Easy examples show that $\text{Num}(M)$ may not contain the spectrum of M (not even the eigenvalues of M contained in L), but it obviously contains an eigenvalue $a \in L$ if there is $u \in L^n$ with $\langle u, u \rangle \in \Delta \setminus \{0\}$ and $Mu = au$. Example 2 shows that it is not sufficient to assume $\langle u, u \rangle \in \Delta_2 \setminus \{0\}$.

We may generalize this observation to two eigenvalues in the following way.

Theorem 1. *Assume that $\langle \cdot, \cdot \rangle$ is definite up to n and take $M \in M_{n,n}(L)$. Fix eigenvalues $a, b \in L$ of M with $a \neq b$ and assume the existence $u, v \in L^n$ such that $\langle u, u \rangle \in \Delta \setminus \{0\}$, $\langle v, v \rangle \in \Delta \setminus \{0\}$, $Mu = au$ and $Mv = bv$.*

- (a) *If $\langle u, v \rangle = 0$, then $\text{Num}(M) \supseteq \{ta + (1-t)b\}_{t \in \Delta \cap (1-\Delta)}$.*
- (b) *If $\langle u, v \rangle \neq 0$, then there is $S \subseteq \text{Num}(M)$ such that $a \in S$, $b \in S$ and S is a Δ_n -ellipse.*

Example 2 shows that we cannot use Δ_2 instead of Δ ; in Example 2 we get a Δ_2 -ellipse $S \subseteq \text{Num}(M)$, but $a \notin S$ and $b \notin S$, because $a \notin \text{Num}(M)$ and $b \notin \text{Num}(M)$.

In section 4 we consider the case in which K as an ordering ([3, Ch. 1]). We prove the following results.

Proposition 1. *Take K with an ordering $<$ such that each positive element of K is a square. Take $L := K(i)$ with σ induced by the map $i \mapsto -i$. Take $M \in M_{n,n}(L)$ such that $M = M^\dagger$. Then $\text{Num}(M)$ is Δ -convex.*

Proposition 2. *Take K with an ordering $<$ such that each positive element of K is a square. Take $L := K(i)$ with σ induced by the map $i \mapsto -i$. Take $M \in M_{n \times n}(K)$ and $a, b \in \text{Num}(M)$ with $a \neq b$. Then there is $S \subseteq \text{Num}(M)$ such that $a \in S$, $b \in S$ and S is either the Δ -convex hull of a and b or an ellipse with two foci containing $\{a, b\}$.*

Theorem 2. *Assume that K is a real closed field and that $L = K(i)$ with σ the complex conjugation. For any $M \in M_{n,n}(L)$ the set $\text{Num}(M)$ is a closed and bounded Δ -convex subset of L .*

As an immediate corollary of Theorem 2 we get the following result.

Corollary 1. *Assume that K is a real closed field and that $L = K(i)$ with σ the complex conjugation. Fix $M, N \in M_{n,n}(L)$ such that $M^\dagger = M$ and $N^\dagger = N$. Then the joint numerical range $\text{Num}(M, N) \subset L^2$ is Δ -convex. If K is real closed, then $\text{Num}(M, N)$ is a closed, bounded and Δ -convex subset of K^2 .*

2. PRELIMINARIES

For any $n > 0$ let $\mathbb{I}_{n \times n}$ denote the identity $n \times n$ matrix (over any field). For any field F set $F^* := F \setminus \{0\}$. Note that $\Delta_2 = \Delta$ if and only if $\Delta_n = \Delta$ for all $n > 1$.

Remark 1. Take $d \in \Delta \setminus \{0\}$ and $u \in L^n$ such that $\langle u, u \rangle = d$. Since $\Delta \setminus \{0\}$ is a multiplicative group, there is $t \in L$ with $t\sigma(t) = 1/d$. Note that $\langle tu, tu \rangle = 1$. Fix $a \in \Delta_n \setminus \{0\}$. By assumption there is $v \in L^n$ such that $\langle v, v \rangle = 0$. Assume the existence of $k \in L$ such that $\langle kv, kv \rangle = 1$. We get $a = 1/c$ with $c := k\sigma(k) \in \Delta \setminus \{0\}$. Since $\Delta \setminus \{0\}$ is a multiplicative group, we get $a \in \Delta$.

Remark 2. For any $M \in M_{n,n}(L)$ and any $u, v \in L^n$ we have $\langle u, Mv \rangle = \langle M^\dagger u, v \rangle = \sigma(\langle v, M^\dagger u \rangle)$. Thus $\text{Num}(M) \subseteq K$ if $M^\dagger = M$.

We fix an element $\beta \in L \setminus K$ with the following property.

First assume $\text{char}(K) \neq 2$. We take as β one of the roots of a polynomial $t^2 - \alpha$, with $\alpha \in K$, α not a square in K and $L \cong K[t]/(t^2 - \alpha)$; note that $\sigma(\beta) = -\beta$ in this case. We have $L = K + K\beta$ as a K -vector space and we see $L^n = K^n + \beta K^n$ as a $2n$ -dimensional K -vector space. For any $z = x + y\beta \in L$, set $\Re z := x$ and $\Im z := y$. We have $x = (z + \sigma(z))/2$ and $y = (z - \sigma(z))/2\beta$. The K -linear maps $\Re : L \rightarrow K$ and $\Im : L \rightarrow K$ depends on the choice of β . Take any $M \in M_{n,n}(L)$ and set $M_+ := (M + M^\dagger)/2$ and $M_- := (M - M^\dagger)/2\beta$. We obvious have $M = M_+ \beta M_-$ and $M_+^\dagger = M_+$. Since $\sigma(1/2\beta) = -1/(2\beta)$ we have $M_-^\dagger = M_-$. Remark 2 gives $\text{Num}(M_+) \subseteq K$ and $\text{Num}(M_-) \subseteq K$. For any $u \in L^n$ we have $\langle u, Mu \rangle = \langle u, M_+ u \rangle + \beta \langle u, M_- u \rangle$. Hence the maps \Re and \Im induces surjections $\text{Num}(M) \rightarrow \text{Num}(M_+)$ and $\text{Num}(M) \rightarrow \text{Num}(M_-)$.

Now assume $\text{char}(K) = 2$. There is $\varepsilon \in K$ such that the polynomial $t^2 + t + \varepsilon$ is irreducible in K , while $L \cong K[t]/(t^2 + t + \varepsilon)$. We take as β one of the roots in L of $t^2 + t + \varepsilon$. Note that $\beta + 1$ is a root of $t^2 + t + \varepsilon$. Thus $\sigma(\beta) = \beta + 1$. We see $L = K + K\beta$ as a 2-dimensional K -vector space and hence L^n as a $2n$ -dimensional K -vector space and $M_{n,n}(L)$ as a $2n^2$ -dimensional K -vector space. If $z = x + y\beta \in L$ with $x, y \in K$, then set $\Re z := x$ and $\Im z := y$. We have $\sigma(z) = x + y + y\beta$ and hence $y = z + \sigma(z)$ and $x = z - \beta z - \beta\sigma(z) = \sigma(\beta)z + \beta\sigma(z)$. The maps $\Re : L \rightarrow K$ and $\Im : L \rightarrow K$ are K -linear. For any $M \in M_{n,n}(L)$ set $M_+ := (\beta + 1)M + \beta M^\dagger$ and $M_- = M + M^\dagger$. Obviously M_- is Hermitian. Since $2\beta = 0$, we have $M = M_+ + \beta M_-$. Since $\sigma(\beta + 1) = \beta$ and $\sigma(\beta) = \beta + 1$, M_+ is Hermitian. Thus the map $z \mapsto \Re z$ (resp. $z \mapsto \Im z$) induces surjections $\text{Num}(M) \rightarrow \text{Num}(M_+) \subseteq K$ (resp. $\text{Num}(M) \rightarrow \text{Num}(M_-) \subseteq K$).

Lemma 1. Assume that $\langle \cdot, \cdot \rangle$ is definite up to n . Take linearly independent $w_1, \dots, w_m \in L^n$, $m \leq n$. Then there are $f_1, \dots, f_n \in L^n$ such that $\langle f_i, f_i \rangle \neq 0$ for all i , $\langle f_i, f_j \rangle = 0$ for all $i \neq j$ and f_1, \dots, f_m span the linear subspace spanned by w_1, \dots, w_m . If $\Delta = \Delta_n$, then we may find f_1, \dots, f_n with the additional condition that $\langle f_i, f_i \rangle = 1$.

Proof. We use induction on m . Call W the linear span of w_1, \dots, w_{m-1} ; if $m-1 > 0$ we take f_1, \dots, f_{m-1} mutually orthogonal and spanning W . Set $W^\perp = \{u \in L^n \mid \langle u, v \rangle = 0 \text{ for all } v \in W\}$. Since W is non-degenerate, we have $\dim W^\perp = n - m + 1$. Since $\langle \cdot, \cdot \rangle$ is definite positive up to n , we have $W \cap W^\perp = \{0\}$, i.e. $L^n = W \oplus W^\perp$ (orthogonal decomposition). Write $f_m = u + v$ with $u \in W$ and $w \in W^\perp$. Since w_1, \dots, w_m are linearly independent, we have $v \neq 0$. Take $f_m = v$. If $m = n$, then we stop. Now assume $m < n$. Let V be the linear span of w_1, \dots, w_m . Set $V^\perp = \{u \in L^n \mid \langle u, v \rangle = 0 \text{ for all } v \in V\}$. We again have $L^n = V \oplus V^\perp$ and we take as f_{m+1} any non-zero element of V^\perp . If $m+1 < n$ we continue using the orthogonal of the linear span of f_1, \dots, f_{m+1} . Now assume $\Delta = \Delta_n$. We assumed exactly the conditions for which the usual Gram-Schmidt orthonormal process works; for instance if f_1, \dots, f_{m-1} are orthonormal, we have $v = w_m - \sum_{i=1}^{m-1} \langle w_m, f_i \rangle f_i$; since $\Delta = \Delta_n$ and $\Delta_n \setminus \{0\}$ is a multiplicative group there is $t \in L$ such that $t\sigma(t) = 1/\langle w, w \rangle$ and we take $f_m := tv$. \square

Lemma 2. Assume $\Delta_2 = \Delta$. Take $e, f \in L$ such that $e \neq f$ and take a, b in the Δ -convex hull S of e, f . Then S contains the Δ -convex hull of a, b .

Proof. Take $t_1, t_2, t \in (\Delta \cap (1 - \Delta))$ such that $a = t_1e + (1 - t_1)f$, $b = t_2e + (1 - t_2)f$. We have $ta + (1 - t)b = [tt_1 + (1 - t)t_2]e + [(1 - t)t_1 + t(1 - t_2)]f$. Since Δ is multiplicative and $\Delta_2 = \Delta$, we have $[tt_1 + (1 - t)t_2] \in \Delta$ and $[(1 - t)t_1 + t(1 - t_2)] \in \Delta$. Hence $[(1 - t)t_1 + t(1 - t_2)] \in (\Delta \cap (1 - \Delta))$. \square

If we need algebraic extensions of L (e.g. because some of the eigenvalues of the matrix M are not in L) the following set-up may be useful. Let K be a perfect field (e.g. assume $\text{char}(K) = 0$). Fix an algebraic closure \overline{L} of L with a fixed inclusion $j : L \hookrightarrow \overline{L}$. The map $j \circ \sigma : L \rightarrow \overline{L}$ extends to a field isomorphism $\sigma' : \overline{L} \rightarrow \overline{L}$ with $\sigma'(x) = x$ for all $x \in K$ ([7, Theorem V.2.8]). We fix one such σ' . For instance, if $K \subseteq \mathbb{R}$, $L = K(i)$ and σ is the complex conjugate, we may just take as σ' the complex conjugate; if $K = \mathbb{F}_q$ and $L = \mathbb{F}_{q^2}$ with σ the Frobenius map $t \mapsto t^q$ we may take as $\sigma' : \overline{\mathbb{F}_{q^2}} \rightarrow \overline{\mathbb{F}_{q^2}}$ the map $t \mapsto t^q$. As seen in the last example σ'^2 may not be the identity (in this example \mathbb{F}_{q^2} is exactly the fixed point set of σ'^2 and σ' has not finite order). We fix one σ' and use it to define the K -bilinear map $\langle \cdot, \cdot \rangle_{\sigma'} : \overline{L}^n \times \overline{L}^n \rightarrow \overline{L}$ by the formula $\langle (u_1, \dots, u_n), (v_1, \dots, v_n) \rangle = \sum_{i=1}^n \sigma'(u_i)v_i$. We say that $\langle \cdot, \cdot \rangle_{\sigma'}$ is definite up to n if $\langle u, u \rangle_{\sigma'} \neq 0$ for all $u \in \overline{L}^n \setminus \{0\}$. This is always the case if $K \subset \mathbb{R}$, $L = K(i)$ and σ and σ' are induced by the complex conjugation.

3. GENERAL RESULTS

Lemma 3. *Assume that either $\text{char}(K) = 0$ or $\text{char}(K) \neq 2$ and there is (w, δ) such that $\delta \in (\Delta \cap (1 - \Delta)) \setminus \{0, 1\}$, $w \in L \setminus K$, $w\sigma(w) = \delta$ and both δ and $1 - \delta$ are squares in K . Take $M \in M_{n,n}(L)$ such that $M = M^\dagger$ and $\sharp(\text{Num}(M)) = 1$. Then $M = c\mathbb{I}_{n \times n}$ for some $c \in K$.*

Proof. If M is Hermitian, then $\text{Num}(M) \subseteq K$ by Remark 2. Assume $\text{Num}(M) = \{c\}$ for some c . If $\text{char}(K) = 0$, then apply [2, Proposition 1]. Hence we may assume the existence of (w, δ) . Since $c \in K$, $M - c\mathbb{I}_{n \times n}$ is Hermitian. Thus taking $M - c\mathbb{I}_{n \times n}$ instead of M we reduce to the case $c = 0$. Write $M = (m_{ij})$, $i, j = 1, \dots, n$. Since $m_{ii} = \langle e_i, Me_i \rangle$, all the diagonal elements of M are zeroes. Assume the existence of $1 \leq i < j \leq n$ with $m_{ij} \neq 0$. We have $m_{ji} = \sigma(m_{ij})$. Take $u = xe_i + ye_j$ with $x\sigma(x) + y\sigma(y) = 1$. Since $m_{ii} = m_{jj} = 0$, we have $0 = \langle u, Mu \rangle = m_{ij}\sigma(x)y + \sigma(m_{ij})x\sigma(y) = 0$. Hence $e + \sigma(e) = 0$, where $e := m_{ij}\sigma(x)y$. Since $\text{char}(K) \neq 2$, there is $\alpha \in K$ such that $e = \alpha\beta$. Take $x, y \in K$ with $x^2 = \delta$ and $y^2 = 1 - \delta$. Since $\delta \notin \{0, 1\}$, we have $xy \neq 0$. Since $e \in K\beta$, we get $m_{ij} = k\beta$ for some $k \in K^*$. Thus $\sigma(x)y \in K$ for all $(x, y) \in L^2$ with $x\sigma(x) + y\sigma(y) = 1$. Taking $x = w$ and $y \in K$ with $y^2 = 1 - \delta$ we get a contradiction. \square

Proposition 3. *Assume that either $\text{char}(K) = 0$ or $\text{char}(K) \neq 2$ and there is (w, δ) such that $\delta \in (\Delta \cap (1 - \Delta)) \setminus \{0, 1\}$, $w \in L \setminus K$, $w\sigma(w) = \delta$ and both δ and $1 - \delta$ are squares in K . A matrix $M \in M_{n,n}(L)$ is Hermitian if and only if $\text{Num}(M) \subseteq K$.*

Proof. If M is Hermitian, then $\text{Num}(M) \subseteq K$ by Remark 2. For an arbitrary $M \in M_{n,n}(L)$ write $M = M_+ + \beta M_-$. M is Hermitian if and only if $M_- = 0\mathbb{I}_{n \times n}$. Since the map $\Im : \text{Num}(M) \rightarrow \Im(M_-)$ is surjective, $\text{Num}(M) \subseteq K$ if and only if $\text{Num}(M_-) = \{0\}$. Apply Lemma 3. \square

Example 1. Take $K = \mathbb{F}_2$, $L = \mathbb{F}_4$ and as σ the Frobenius map $t \rightarrow t^2$. If $u = (x, y) \in L^2$ we have $\langle u, u \rangle = 1$ if and only if $x^3 + y^3 = 1$, i.e. (since $z^3 = 1$ if $z \in \mathbb{F}_4^*$) if and only if $(x, y) \in \{(0, 1), (1, 0)\}$. Hence for each $M \in M_{2,2}(\mathbb{F}_4)$ the numerical range $\text{Num}(M)$ is the set of all diagonal elements of M . Hence there are 16 matrices $M \in M_{2,2}(\mathbb{F}_4)$ with $\text{Num}(M) = \{0\}$ and 4 of them are Hermitian. This is the only example we know (and the only one for finite fields).

Proof of Theorem 1: Taking a multiple of u and v instead of u and v and applying Remark 1 we reduce to the case $\langle u, u \rangle = \langle v, v \rangle = 1$. Since $a \neq b$, u and v are linearly independent. Taking $\frac{1}{b-a}(M - a\mathbb{I}_{n \times n})$ instead of M we reduce to the case $a = 0$ and $b = 1$. Thus $u \neq 0$, $Mu = 0$ and $Mv = v$.

(a) First assume $\langle u, v \rangle = 0$ and hence $\langle v, u \rangle = 0$. Fix $\delta \in \Delta \cap (1 - \Delta)$ and write $\delta = y\sigma(y)$ and $1 - \delta = x\sigma(x)$ for some $x, y \in L$. Take $m = xu + yv$. We have $\langle m, m \rangle = 1$, because $x\sigma(x) + y\sigma(y) = 1$. We have $Mm = yv$ and hence $\langle m, Mm \rangle = \sigma(y)y = \delta$.

(b) Now assume $d := \langle u, v \rangle \neq 0$. Set $w := v - \langle v, u \rangle u$. Since u, w and u, v have the same linear span, we have $w \neq 0$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate up to n , we have $c := \langle w, w \rangle \in \Delta_n \setminus \{0\}$. We have $\langle w, u \rangle = 0$. Take $m = xu + yw$. We have $\langle m, m \rangle = 1$ if and only if $\sigma(x)x + cy\sigma(y) = 1$. We have $Mm = yv = yw + ydu$ and hence $\langle m, Mm \rangle = d^2\sigma(x)y + c\sigma(y)y$. The latter set is a Δ_n -ellipse with 2 foci. \square

Example 2. Assume that $\langle \cdot, \cdot \rangle$ is definite up to 2, but $\Delta_2 \neq \Delta$ and fix $\delta \in \Delta_2 \setminus \Delta$. We may take $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$ and σ the complex conjugate ([2, Example 2]). Note that $\delta \neq 0$. Fix $u \in L^2$ such that $\langle u, u \rangle = \delta$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate, there is $m \in L^2$ such that $\langle m, u \rangle = 0$ and $m \neq 0$. Let $M \in M_{2,2}(L)$ be the only matrix with $Mu = 0$ and $Mm = m$. Hence 0 and 1 are the eigenvalues of M . Set $\delta_1 := \langle m, m \rangle$. We claim that $0 \notin \text{Num}(M)$. Take $v = xm + yu$. Since $Mv = xm$ and m, u are orthogonal, we have $\langle v, Mv \rangle = 0$ if and only if $x = 0$. We have $\langle v, v \rangle = \sigma(x)x\delta \neq 1$ (Remark 1). Now we check that $1 \notin \text{Num}(M)$. We have $\langle v, v \rangle = 1$ if and only if $\delta_1 x\sigma(x) + \delta y\sigma(y) = 1$. We have $\langle v, Mv \rangle = 1$ if and only if $\delta_1 x\sigma(x) = 1$, i.e. if and only if $y = 0$. Hence $1 \in \text{Num}(M)$ if and only if there is $x \in L$ with $\delta_1 x\sigma(x) = 1$, i.e. if and only if $\langle f_1, f_1 \rangle = 1$, where $f_1 = xm$. Write $f_1 = g_1 e_1 + g_2 e_2$ and set $f_2 := g_2 e_1 + g_1 e_2$. We get $\langle f_2, f_2 \rangle = 1$ and $\langle f_2, f_1 \rangle = 0$. Hence there is $t \in L^*$ with $u = t f_2$. We get $\delta = t\sigma(t) \in \delta$, a contradiction. This situation is satisfied if we take $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$, σ and σ' the complex conjugate. Note that in this case σ'^2 is the identity and $\langle \cdot, \cdot \rangle_{\sigma'}$ is definite for all n . Hence we do not see a reasonable way of weakening the assumptions in Theorem 1.

Lemma 4. Fix an integer $n \geq 2$ and assume that $\langle \cdot, \cdot \rangle_{\sigma'}$ is definite up to n and take $M \in M_{n,n}(L)$ such that $M^\dagger = M$. Let $c \in \overline{L}$ be any eigenvalue of M . Then $\sigma'(c) = c$.

Proof. Take $u \in \overline{L}^n$, $u \neq 0$, such that $Mu = cu$. We have $c\langle u, u \rangle = \langle u, Mu \rangle = \langle Mu, u \rangle = \sigma'(c)\langle u, u \rangle$. \square

Lemma 5. Fix an integer $n \geq 2$ and assume that $\langle \cdot, \cdot \rangle_{\sigma'}$ is definite up to n and take $M \in M_{n,n}(L)$ such that $M^\dagger = M$. Take eigenvalues $c, d \in \overline{L}$ of M such that $c \neq d$ and any $u, v \in \overline{L}$ such that $Mu = cu$ and $Mv = dv$. Then $\langle u, v \rangle_{\sigma'} = 0$.

Proof. We have $d\langle u, v \rangle_{\sigma'} = \langle u, Mv \rangle_{\sigma'} = \langle Mu, v \rangle_{\sigma'} = \sigma'(c)\langle u, v \rangle_{\sigma'}$. Lemma 4 gives $\sigma'(c) = c \neq d$. \square

Remark 3. Take $A \in M_{n,n}(L)$, $B \in M_{m,m}(L)$ and set $M := A \oplus B \in M_{n+m,n+m}(L)$. $\text{Num}(M)$ is the Δ -convex hull of $\text{Num}(A)$ and $\text{Num}(B)$ ([2, Lemma 7]).

Lemma 6. Take $n = 2$ and fix $b \in L^*$. Take

$$M = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

Then $\text{Num}(M)$ is the ellipse with foci at $\{0, 1\}$ and parameters $(\delta_1, \delta_2, d_1, d_2) = (1, 1, b, 1)$.

Proof. Take $u = xe_1 + ye_2$. We have $\langle u, u \rangle = 1$ if and only if $x\sigma(x) + y\sigma(y) = 1$ and $\langle u, Mu \rangle = bx\sigma(y) + y\sigma(y)$. \square

Lemma 7. Take $n = 2$ and the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $\text{Num}(M)$ is the ellipse with one focus at 0 and $\delta_1 = \delta_2 = 1$.

Proof. Take $u = xe_1 + ye_2$. We have $\langle u, u \rangle = 1$ if and only if $x\sigma(x) + y\sigma(y) = 1$ and $\langle u, Mu \rangle = \sigma(x)y$. \square

Proposition 4. Assume that $\langle \cdot, \cdot \rangle$ is definite up to 2 and take $M \in M_{2,2}(L)$ with two different eigenvalues, both in L . Assume either $\Delta_2 = \Delta$ or that M has at least one eigenvector u with $\langle u, u \rangle \in \Delta$. Then $\text{Num}(M)$ is an ellipse with two foci or the Δ -convex hull of its eigenvalues.

Proof. If $\Delta_2 = \Delta$, then $\langle w, w \rangle \in \Delta \setminus \{0\}$ for any $w \in L^2 \setminus \{0\}$, because $\langle \cdot, \cdot \rangle$ is definite up to 2. Call c_1 and c_2 the two eigenvalues of M with c_1 with eigenvalue u with $\langle u, u \rangle \in \Delta \setminus \{0\}$. Take $t \in L^*$ such that $t\sigma(t) = 1/\langle u, u \rangle$ and set $f_1 := tu$. Write $u = a_1e_1 + a_2e_2$ with $a_i \in L$ and set $f_2 := a_2e_1 - a_1e_2$. We have $\langle f_i, f_j \rangle = 0$ for all $i \neq j$. We have $\langle f_2, f_2 \rangle = \langle f_1, f_1 \rangle = 1$. Taking $\frac{1}{c_2 - c_1}(M - c_1\mathbb{I}_{2 \times 2})$ instead of M we reduce to the case $c_1 = 0$ and $c_2 = 1$. Since f_1 and f_2 is an orthonormal basis, we may use it instead of e_1 and e_2 to compute $\text{Num}(M)$. If f_2 is an eigenvector of M , then it has eigenvalue 1 and hence M is unitarily equivalent to $0\mathbb{I}_{1 \times 1} \oplus \mathbb{I}_{1 \times 1}$ and hence $\text{Num}(M) = \Delta \cap (1 - \Delta)$ (the Δ -convex hull of 0 and 1). \square

Proposition 5. Assume either $\text{char}(L) \neq 2$ or that L is perfect. Assume that $\langle \cdot, \cdot \rangle$ is definite up to 2 and take $M \in M_{2,2}(L)$ with a unique eigenvalue, c . Then $c \in L$. Assume also that either $\Delta_2 = \Delta$ or that M has an eigenvector u with $\langle u, u \rangle \in \Delta$. Then either $M = c\mathbb{I}_{2 \times 2}$ and $\text{Num}(M) = \{c\}$ or $\text{Num}(M)$ is an ellipse with one focus.

Proof. The characteristic polynomial of M has degree 2 and c is its only root over \overline{L} . We have $c \in L$, because in the case $\text{char}(L) = 2$ we assumed that L is perfect. Taking $M - c\mathbb{I}_{2 \times 2}$ instead of M we reduce to the case $c = 0$. By assumption M has an eigenvector u with $\langle u, u \rangle \in \Delta \setminus \{0\}$. Take $t \in L^*$ such that $t\sigma(t) = 1/\langle u, u \rangle$ and set $f_1 := tu$. Write $u = a_1e_1 + a_2e_2$ with $a_i \in L$ and set $f_2 := a_2e_1 - a_1e_2$. We have $\langle f_i, f_j \rangle = 0$ for all $i \neq j$. We have $\langle f_2, f_2 \rangle = \langle f_1, f_1 \rangle = 1$. Write $M = (a_{ij})$, $i, j = 1, 2$, in the orthonormal basis f_1 and f_2 . We have $a_{11} = a_{21} = 0$. Since 0 is the only eigenvalue of M , we have $a_{22} = 0$. If f_2 is an eigenvector of M , then $M = 0\mathbb{I}_{2 \times 2}$. Hence we may assume that $a_{12} \neq 0$. Apply Lemma 7 to $(1/a_{12})M$. \square

4. ORDERED FIELDS

Let $(K, <)$ be any ordered field. Fix a real closure $\mathcal{R} \supseteq K$ ([3, Theorem 1.3.2]) and call again $<$ the ordering of \mathcal{R} extending the ordering $<$ of K . Set $K_0 := K$. For each $i \geq 1$ let $K_i \subseteq \mathcal{R}$ be the field obtained adding to K_{i-1} all the square-roots of the positive (with respect to the ordering $<$ of \mathcal{R}) elements of K_{i-1} . Set $\tilde{K} := \cup K_{i \geq 1}$. \tilde{K} is the smallest subfield of \mathcal{R} such that $K \subseteq \tilde{K}$ and all positive elements of \tilde{K} are square ([6, Proposition 16.4]; ordered fields with this property are called *euclidean* in [6, Proposition 16.2]). Set $\tilde{L} = \tilde{K}(i)$ and use the complex conjugation σ' to get a positive definite sesquilinear form up to any n . Since each square is contained in Δ , (\tilde{L}, σ) has $\Delta = \tilde{K}_{\geq 0} \supseteq \Delta_n$ and so $\Delta = \Delta_n$ for all n .

Proof of Proposition 1: If $z = x + yi \in L$ with $x, y \in K$, then $\langle z, z \rangle = x^2 + y^2 \geq 0$ with equality if and only if $x = y = 0$. Hence $\langle \cdot, \cdot \rangle$ is definite up to any $m > 0$. Since Δ contains all squares, we have $\Delta_m = \Delta$ for all $m \geq 2$. If $\text{Num}(M) = \{c\}$ for some $c \in K$, then $M = c\mathbb{I}_{n \times n}$ by [2, Proposition 1]. Now assume the existence of $a, b \in \text{Num}(M)$ such that $a \neq b$ and take $u, v \in L^n$ such that $\langle u, u \rangle = \langle v, v \rangle = 1$, $\langle u, Mu \rangle = a$ and $\langle v, Mv \rangle = b$. By Lemma 1 we may assume that $u = e_1$ and $v = e_2$. Set $N := M|_{(Le_1 + Le_2)}$. We have $a, b \in \text{Num}(N)$ and $\text{Num}(N) \subseteq \text{Num}(M)$. Write $N = (a_{ij})$, $i, j = 1, 2$. We have $a_{11} = a \in K$, $a_{22} = b \in K$, $a_{21} = \langle v, Mu \rangle = \langle Mv, u \rangle = \sigma(\langle u, Mv \rangle) = \sigma(a_{12})$. Hence $N^\dagger = N$. Write $a_{12} = x + yi$ with $x, y \in K$. Let $f(t)$ be the characteristic polynomial of N . Taking $N - \frac{a_{11} + a_{22}}{2}\mathbb{I}_{2 \times 2}$ instead of N we reduce to the case $a_{11} = -a_{22}$, i.e. N has trace 0, i.e. $f(t) = t^2 - a_{11}^2 - a_{12}\sigma(a_{12}) = t^2 - a_{11}^2 - x^2 - y^2$. Since sums of squares of elements of K are squares by our assumption on K , we have $f(t) = t^2 - d^2$ for some $d \in K$. Hence all the eigenvalues of N are contained in K . If $d \neq 0$, then N has two different eigenvalues, d and $-d$, and Lemma 5 gives that N is unitarily equivalent to $(-d)\mathbb{I}_{1 \times 1} \oplus d\mathbb{I}_{1 \times 1}$. In this case we apply Remark 3. If $d = 0$, then $a_{11} = x = y = 0$ and so $N = 0\mathbb{I}_{2 \times 2}$. Thus $\text{Num}(N) = \{0\}$, contradicting the assumption $a \neq b$. \square

Proposition 6. *Take K with an ordering $<$ such that each positive element of K is a square. Take $L := K(i)$ with σ induced by the map $i \mapsto -i$. Take $M \in M_{2,2}(K)$. Then $\text{Num}(M)$ is either a point, or a Δ -segment or a Δ -ellipse with one or 2 foci.*

Proof. Write $M = (m_{ij})$, $i, j = 1, 2$. Using $M - \frac{m_{11} + m_{22}}{2}\mathbb{I}_{2 \times 2}$ instead of M we reduce to the case $m_{11} + m_{12} = 0$. Hence the characteristic polynomial $f(t)$ of M is of the form $f(t) = t^2 + d$ for some $d \in K$. If $d \leq 0$ (resp. $d > 0$), then it has 2 roots in K (resp. L), because every positive element of K has a square root in K . If $d \neq 0$, then we apply Proposition 4. If $d = 0$, then either $M = 0\mathbb{I}_{2 \times 2}$ or we apply Proposition 5. \square

Proof of Proposition 2: Let \mathcal{R} be a real closure of $(K, <)$. Since $(K, <)$ is euclidean, we have $\Delta = \Delta_m$ for every $m > 0$ and $\langle \cdot, \cdot \rangle_{\sigma'}$ is definite up to any $m > 0$. Take $u, v \in L^n$ such that $\langle u, u \rangle = \langle v, v \rangle = 1$, $a = \langle u, Mu \rangle$ and $b = \langle v, Mv \rangle$. Since $a \neq b$, u and v are not proportional. Let W be the linear span u and v . By Lemma 1 we may assume $n = 2$ with the matrix $N := M|_W$. If N has a unique eigenvalue, c , then we apply Proposition 5. If N has two different eigenvalues, both of them in L , we use Proposition 5. Now assume that N has two eigenvalues $c_i \in \mathcal{R}(i)$, $i = 1, 2$, none of them in L . Since the characteristic polynomial of N has coefficients in K ,

these eigenvalues generate a degree 2 Galois extension K_1 of K , i.e. there are a non-square $m \in K$ such that $K_1(g)$ with $g^2 = e$. Since $(K, <)$ is euclidean, $e < 0$ and $-e = f^2$ for some $f \in K$. Thus $K_1 = K(i) = L$, a contradiction. \square

If K is real closed, then Proposition 2 is trivial, because in that case $\text{Num}(M)$ is a semi-algebraically connect bounded and closed semi-algebraic subset of K (see Example 4), i.e. a closed interval ([3, Proposition 2.1.7]).

Remark 4. Take as K a real closed ordered field ([3, §1.2]) and take $L = K(i)$ with σ the complex conjugation. Thus L is algebraically closed, $\sigma' = \sigma$ and σ'^2 is the identity. Therefore for every $n > 0$ $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\sigma'}$ are definite up to n . Since $K_{\geq 0}$ is the set of all squares of elements of K , we have $\Delta_n \subseteq K_{\geq 0} = \Delta$ and so $\Delta_n = \Delta$ for all n . The set $S^{2n-1} := \{u \in L^n \mid \langle u, u \rangle = 1\}$ is a closed and bounded subset of $L^n = K^{2n}$ for the euclidean topology and the map $u \mapsto \langle u, Mu \rangle$ is real algebraic. Thus $\text{Num}(M)$ is a closed and bounded semi-algebraic subset of L ([3, Proposition 2.5.7]). Since the sphere S^{2n-1} is semi-algebraically connected in the sense of [3, Definition 2.4.2], $\text{Num}(M)$ is semi-algebraically connected. By [3, Proposition 2.5.13] $\text{Num}(M)$ is semi-algebraically path connected. Take $a, b \in L$ with $a \neq b$. The Δ -convex hull $A \subseteq L$ is the segment $\{ta + (1-t)a\}_{t \in K, 0 \leq t \leq 1}$. Hence Δ -convexity is preserved by any K -affine map. Since L is algebraically closed, to prove that $\text{Num}(M)$ is convex, it is sufficient to prove that ellipse with one or two foci are Δ -convex, but we prefer to follow the convexity proof given in [9]. To show that ellipses with one foci are disks of $L = K^2$ (resp. an ellipse with two foci plus its interior in the sense of the euclidean topology) we may use the following remarks. First of all we reduce to the case $\delta_1 = \delta_2 = 1$. Under this assumption they are the numerical range of the matrix M appearing in Lemma 7 (resp. Lemma 6). On $L \setminus \{0\}$ the function $|z|$ is semi-algebraic and so $\Re(z/|z|)$ and $\Im(z/|z|)$. We use this functions instead of the functions \cos and \sin to show that $\{|z| \leq 1/2\} = \text{Num}(M)$ in the case of Lemma 7 following the proof in [5, Example 1, page 1]. To adapt the proof of [5, Example 2, pages 2,3 and Lemma 1.1] to get the Δ -convexity in the set-up of Lemma 6 we need to make sense of some trigonometric expression like \sin and \cos ; if $K \neq \mathbb{R}$ we cannot use the real exponential function to get the trigonometric function ([1]). With our substitute of the functions \cos and \sin , the first one is even and the second one is odd. For any fixed $z \in S^1 := \{|z| = 1\}$ there are $a, b \in S^1$ with $a^2 = z$ and $b^3 = z$. Instead of $\cos(u+v)$ (resp. $\sin(u+v)$) we may use $\cos(u)\cos(v) - \sin(u)\sin(v)$ (resp. $\sin(u)\cos(v) + \cos(u)\sin(v)$).

Theorem 2 easily follows from the following result ([9, Lemma 2]; the proof in [9, page 4] works with minimal modifications).

Proposition 7. *Assume that K is a real closed field and that $L = K(i)$ with σ the complex conjugation. Fix $M \in M_{n,n}(L)$ such that $M^\dagger = M$ and any $t \in L$. Then either $\nu_M^{-1}(t) = \emptyset$ or $\nu_M^{-1}(t)$ is semi-algebraically arc-connected.*

Proof. If $t \notin K$, then $\nu_M^{-1}(t) = \emptyset$ by Lemma 4 and hence we may assume $t \in K$. If $t \in K$, then $M - t\mathbb{I}_{n \times n}$ is Hermitian. Since $\text{Num}(M - t\mathbb{I}_{n \times n}) = \text{Num}(M) - t$, taking $M - t\mathbb{I}_{n \times n}$ instead of M we reduce to the case $t = 0$. Since UMU^\dagger is Hermitian and $\text{Num}(UMU^\dagger) = \text{Num}(M)$ for every unitary U , we reduce to the case in which M is diagonal, say $M = (m_{ij})$ with $m_{ii} \in K$ and $m_{ij} = 0$ for all $i \neq j$. Take $a = (a_1, \dots, a_n) \in \nu_M^{-1}(0)$. For all $(z_1, \dots, z_n) \in L^n$ with $\sigma(z_i)z_i = 1$ for all i we have $(z_1 a_1, \dots, z_n a_n) \in \nu_M^{-1}(0)$, because M is a diagonal matrix. Since the circle

$S^1 = \{\sigma(z)z = 1\} \subset L$ is semi-algebraically arc-connected, it is sufficient to prove that if $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \nu_M^{-1}(0)$ with $a_i, b_i \in K$ for all i , then a and b are connected by a semi-algebraic arc contained in $\nu_M^{-1}(0)$. As in [9, page 4] it is sufficient to use the semi-algebraic arc $u(t) = (u_1(t), \dots, u_n(t))$ from the unit interval $[0, 1]$ into $\nu_M^{-1}(0) \cap K^n$ with $u_j(t) = \sqrt{(1-t)a_j^2 + tb_j^2}$. \square

Proof of Theorem 2: ([9, page 4]) Fix $a, b \in \text{Num}(M)$ such that $a \neq b$. Taking $\frac{1}{b-a}(M - a\mathbb{I}_{n \times n})$ instead of M we reduce to the case $a = 0$ and $b = 1$. Since Δ is the set of all non-negative elements of K , it is sufficient to prove that $\text{Num}(M)$ contains the closed interval $[0, 1] \subset K$. Take $u, v \in L^n$ such that $\langle u, u \rangle = \langle v, v \rangle = 1$, $\langle u, Mu \rangle = 0$ and $\langle v, Mv \rangle = 1$. Since $\{0, 1\} \in K$, we have $\langle u, M_+u \rangle = 0$, $\langle u, M_-u \rangle = 0$, $\langle v, M_+v \rangle = 1$ and $\langle v, M_-v \rangle = 0$. Since M_- is Hermitian, and $\{u, v\} \subset \nu_{M_-}^{-1}(0)$, there is a semi-algebraic arc $m : [0, 1] \rightarrow \nu_{M_-}^{-1}(0)$ with $m(0) = u$ and $m(1) = v$ (Proposition 7). We have $\langle m(t), Mm(t) \rangle = \langle m(t), M_+m(t) \rangle$, because $\langle m(t), M_-m(t) \rangle = 0$ for all $t \in [0, 1]$. Since M_+ is Hermitian, $\langle m(t), M_+m(t) \rangle \in K$ for all K . The set Z of all $\langle m(t), M_+m(t) \rangle$, $t \in [0, 1]$, is a semi-algebraic arc contained in K and hence it is an interval ([3, Proposition 2.1.7]). Since $\{0, 1\} \subset Z \subseteq \text{Num}(M)$, then $[0, 1] \subseteq \text{Num}(M)$. \square

Proof of Corollary 1: Identify L^2 with $K^2 + iK^2$ sending any $z = (z_1, z_2) \in L^2$ to $(\Re z_1, \Re z_2, \Im z_1, \Im z_2)$ and then identify any $(a_1, a_2) \in K^2$ with $a_1 + ia_2 \in L$. Take $A := M + iN \in M_{n,n}(L)$. Since M and N are Hermitian, we have $A_+ = M$, $A_- = N$, $\text{Num}(M) \subset K$, $\text{Num}(N) \subset K$ and $c \in \text{Num}(A)$ if and only if $\Re c \in \text{Num}(M)$ and $\Im c \in \text{Num}(N)$. Apply Theorem 2 to A . \square

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